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SELF-SIMILAR SOLUTIONS OF THE SECOND ORDER BOUNDARY LAYER OF AN INCOMPRESSIBLE FLUID WITH HEAT TRANSFER

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Abstract—Self-similar solutions of the second-order incompressible boundary-layer equations, containing terms due to longitudinal curvature, transverse curvature, external vorticity, displacement speed and stagnation enthalpy gradients have been studied. Numerical as well as several closed form solutions are presented for both the skin friction and heat transfer rate at the wall.

NOMENCLATURE

- $B_1(\Psi_1)$, Bernoulli function defined as $= P_1 + (U_1^2 + V_1^2)/2$;
- D, function defined by (25d);
- f, g, dimensionless velocity in s-direction and temperature due to first-order boundary layer;
- f_1, f_2, g_1, g_2, functions defined by equation (28a) and (28b);
- F', G, corrections to f' and g due to second-order boundary layer;
- F_1, F_2, G_1, G_2, functions defined by equations (29a) and (29b);
- h_1, h_2, first and the second-order local stagnation enthalpies in inner flow defined by $h_1 = t_1 + m^2u_1^2/2$ and $h_2 = t_2 + m^2 u_1 u_2$
- $H_1(\Psi_1)$, stagnation enthalpy function defined as $= T_1 + m^2 (U_1^2 + V_1^2)/2$;
- j, a number defined as zero for two dimensional flow and unity for axisymmetric flow;
- k_1, longitudinal curvature parameter defined by equation (25b);
- k_t, transverse curvature parameter defined by (25c)
- K, longitudinal surface curvature of the body;
- l, characteristic length of the body;
- $L(,)$, differential operator defined by equation (36);
- m^2, a parameter defined as $= U_r^2/C_p t_r$;
- $M(,)$, differential operator defined by equation (37);
- n, coordinate normal to body;
- N, stretched (Prandtl) normal coordinate defined as $= R^{\frac{1}{2}} n$;
- P_1, P_2, first- and second-order static pressures in outer flow;
- q, heat-transfer rate given by (51);
- R, characteristic Reynolds number of the flow defined as $= U_r \rho l/\mu$;
- s, coordinate along the body;
- t, static temperature;
- t_1, t_2, first- and second-order temperatures in inner flow;
- T_1, T_2, first- and second-order temperatures in outer flow;
- u_1, u_2, first- and the second order velocities in s-direction in inner flow;
- U_1, U_2, first- and second-order outer flow velocities in s-direction;
- V_1, V_2, first- and the second-order outer flow velocity in n-direction;
- U, free stream velocity;
- v_1, v_2, first- and the second-order velocity in n-direction in inner flow;

X, function defined by (25e);
Z, function defined by (25f).

Greek symbols

α, displacement thickness due to first order boundary layer defined by (25a);
α_0, α_1, functions defined by $\alpha = \sum_{i=0} \alpha_i \xi^{mi}$;
β_0, β_1, functions defined by equation (26a);
η, boundary-layer variable defined by (11);
θ, angle between axis of axisymmetric body and the tangent to meridian curve at any point (see Fig. 1);
Λ_D, principal displacement speed function defined by equation (24);
$\Lambda_{d_0}, \Lambda_{d_1}$, functions defined by equation (27c);
Λ_{k_l}, principal longitudinal curvature function defined by (24);
$\Lambda_{l_0}, \Lambda_{l_1}$, functions defined by equation (27a);
Λ_{k_t}, principal transverse curvature functions defined by (24);
$\Lambda_{t_0}, \Lambda_{t_1}$, functions defined by equation (27b);
Λ_{U_1}, principal velocity function defined by (24);
$\Lambda_{t_w - H_1}$, principal thermal function defined by (24);
Λ_0, Λ_1, functions defined by equations (26b);
μ, viscosity of fluid;
ξ, boundary-layer variable defined by (11);
ρ, density of fluid;
σ, Prandtl number of the fluid;
τ, skin friction given by (49);
ψ_1, ψ_2, stream functions due to first- and second-order boundary layers defined by equations (12) and (13);
Ψ_1, Ψ_2, first- and second-order stream functions of outer flow.

Superscripts

$'$, differentiation with respect to variable η;
d, displacement speed;
H, stagnation enthalpy;
l, longitudinal curvature;
t, transverse curvature;
v, vorticity.

Subscripts

∞, free stream;
N, partial derivative with respect to N;
r, characteristic reference values;
s, partial derivative with respect to s;
ξ, partial derivative with respect to ξ.

1. INTRODUCTION

THE CLASSICAL boundary-layer theory of Prandtl represents the leading term (let us call it the first-order) in an asymptotic expansion of Navier–Stokes equations for large Reynolds numbers. By now it is well known, e.g. [1, 2] that when a systematic approach is made to calculate the second term in such asymptotic expansion, the effects of curvature, external vorticity, stagnation enthalpy gradient are contained in this term, provided that the parameters representing these qualities—curvature etc.—are of the order unity. The object of the present work is to calculate the second-order effects in two dimensional and axisymmetric incompressible flows.

The theory for the study of such effects was formulated by Van Dyke [1] and applied to a few special cases [2, 3] using the well-known Blasius series method. The shortcomings of the Blasius series method, however, are well known [4]. The first author in his thesis [5] has applied the Görtler [6] method to the second-order momentum and heat-transfer problem and studied few higher order terms in the Görtler series by introducing three parameters, i.e. principal longitudinal curvature function, principal transverse curvature function and the principal displacement speed function. These principal functions are of fundamental importance in determining the structure of the second-order effects. Since the completion of the present investigation [5], Werle and Davis [7] have published self-similar solutions for the second-order momentum transfer. The main differences between the present work and [7] will be pointed out in Section 5.

2. BOUNDARY-LAYER EQUATIONS

The governing equations for the second-order boundary layer theory are obtained from Navier–Stokes equations by the method of matched asymptotic expansions with a perturbation parameter $R^{-\frac{1}{2}}$. This method results in replacing the Navier–Stokes equations by two separate sets of equations, one set which is valid in the region near the wall whose thickness is of the order $R^{-\frac{1}{2}}$ and the other set out side this region. The solutions of these two regions are matched in an "overlap" domain. The governing equations for the first and the second-order boundary-layer flows of an incompressible

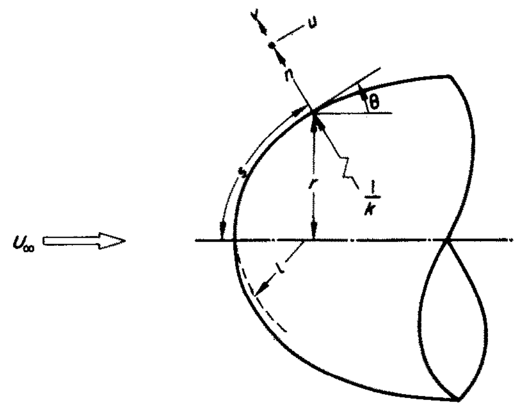


FIG. 1. Coordinate system.

fluid with small temperature changes (dissipation is neglected; this being easily justified) in the coordinate system shown in Fig. 1 are as follows [1]

First-order boundary-layer equations
Continuity

$$(r^j u_1)_s + (r^j v_1)_N = 0 \tag{1}$$

Momentum

$$u_1 u_{1s} + v_1 u_{1N} - u_{1NN} = U_1(s,0) U_{1s}(s,0) \tag{2}$$

Energy

$$u_1 h_{1s} + v_1 h_{1N} - \sigma^{-1} h_{1NN} = 0 \tag{3}$$

Boundary conditions

$$u_1(s,0) = 0 = v_1(s,0), \quad h_1(s,0) = t_w(s) \tag{4a, 4b, 4c}$$

Matching conditions:

$$\left.\begin{aligned} u_1(s,N) &= U_1(s,0) \\ h_1(s,N) &= H_1(0) \end{aligned}\right] N \to \infty \qquad \begin{aligned}(5a)\\(5b)\end{aligned}$$

Second-order boundary-layer equations
Continuity:

$$(r^j u_2)_s + (r^j v_2)_N = -Kr^j (N v_1)_N - j(N u_1 r^j \cos\theta/r)_s - j(N v_1 r^j \cos\theta/r)_N \tag{6}$$

Momentum:

$$\begin{aligned} u_1 u_{2s} + v_1 u_{2N} + u_2 u_{1s} + v_2 u_{1N} - u_{2NN} &= u_{1N} j \cos\theta/r + K[N u_1 u_{1s} \\ &\quad - N U_1(s,0) U_{1s}(s,0) + u_{1N} - u_1 v_1] \\ &\quad - \{K U_1^2(s,0) + K \int_N^\infty [U_1^2(s,0) - u_1^2]\, \mathrm{d}N\}_s \\ &\quad - r^j B_1(0) \lim_{N\to\infty} (\Psi_{1s} - N\Psi_{1Ns}) + [U_1(s,0) U_2(s,0)]_s \end{aligned} \tag{7}$$

Energy:

$$u_1 h_{2s} + v_1 h_{2N} + u_2 h_{1s} + v_2 h_{1N} - \sigma^{-1} h_{2NN} = K(N u_1 h_{1s} + \sigma^{-1} h_{1N}) + \sigma^{-1} h_{1N} j \cos\theta/r \tag{8}$$

Boundary conditions:

$$u_2(s,0) = 0 = v_2(s,0), h_2(s,0) = 0 \tag{9a, 9b, 9c}$$

Matching conditions:

$$u_2(s,N) = -KNU_1(s,0) + r^j N B_1'(0) + U_2(s,0), \quad N \to \infty \tag{10a}$$

$$h_2(s,N) = H_1'(0)\Psi_1(s,N), \quad N \to \infty. \tag{10b}$$

3. LOCALLY-SIMILAR EQUATIONS

In order to study the boundary-layer flows one generally resorts to similar solutions (i.e. the partial differential equations are reduced to total equations). However, in majority of engineering problems with diverse applications, it is a rare occurrence indeed when all the conditions for similarity are satisfied. In the literature various methods have been used to study the non-similar flows. Relative merits and demerits of these methods are discussed in [5] with the conclusion that Görtler [6] type variables defined by

$$\xi = \int_0^s U_1(s,0)\, r^{2j}\, ds, \quad \eta = \frac{r^j U_1(s,0)}{\sqrt{(2\xi)}} N \quad (11)$$

are advantageous.

Defining the first- and second-order stream functions ψ_1 and ψ_2 by

$$r^j u_1 = \psi_{1N}, r^j v_1 = -\psi_{1s} \quad (12a, b)$$

$$r^j[u_2 + jNu_1 \cos\theta/r] = \psi_{2N} \quad (13a)$$

$$r^j[v_2 + (K + j\cos\theta/r)Nv_1] = -\psi_{2s} \quad (13b)$$

the equations of continuity (1) and (6) are automatically satisfied and the rest of the equations with the help of the transformations

$$\psi_1 = (2\xi)^{\frac{1}{2}} f(\xi, \eta) \quad (14a)$$

$$h_1 = [t_w - H_1(0)]\, g(\xi, \eta) + H_1(0) \quad (14b)$$

$$\psi_2 = (2\xi)^{\frac{1}{2}} F(\xi, \eta) \quad (15a)$$

$$h_2 = [t_w - H_1(0)]\, G(\xi, \eta) \quad (15b)$$

reduce to the following equations.

First-order boundary-layer equations

$$f''' + ff'' + \Lambda_{U_1}(1 - f'^2) - 2\xi(f' f'_\xi - f_\xi f'') = 0 \quad (16)$$

$$\sigma^{-1} g'' + fg' - f' g \Lambda_{t_w - H_1} - 2\xi(f' g_\xi - f_\xi g') = 0 \quad (17)$$

$$f(\xi, 0) + 2\xi f_\xi(\xi, 0) = 0 \quad (18a)$$

$$f'(\xi, 0) = 0, \quad f'(\xi, \infty) = 1, \quad (18b, c)$$

$$g(\xi, 0) = 1, g(\xi, \infty) = 0. \quad (19a, b)$$

Second-order boundary-layer equations

$$F''' + fF'' - 2\Lambda_{U_1} f'F' + f''F - 2\xi(F'f'_\xi - F_\xi f'' + f'F'_\xi - f_\xi F'') = k_1[-\eta(1 + \Lambda_{U_1}) \times f''' + (\Lambda_{k_1} + \Lambda_{U_1} - 1)(f'' + ff' + 2\xi f' f_\xi + 4\xi \int_\eta^\infty f' f'_\xi\, d\eta) + (2\Lambda_{U_1} + \Lambda_{k_1})(\Lambda_{U_1}\eta + \alpha + 2\xi\alpha_\xi)]/(1 + \Lambda_{U_1}) + k_t[-\eta(2\Lambda_{U_1} + f''') + f'' + ff' - \Lambda_{k_t}\eta f'^2 + 2\xi f' f_\xi] + X(\alpha + 2\xi\alpha_\xi) - D(2\Lambda_{U_1} + \Lambda_D), \quad (20)$$

$$\sigma^{-1} G'' + fG' - f' G \Lambda_{t_w - H_1} - 2\xi(f' G_\xi - f_\xi G') = \Lambda_{t_w - H_1} F' g - F g' + 2\xi(F' g_\xi - F_\xi g') - \sigma^{-1}(k_1 + k_t)(\eta g')', \quad (21)$$

$$F(\xi, 0) + 2\xi F_\xi(\xi, 0) = 0, \quad F'(\xi, 0) = 0, \quad (22a, b)$$

$$F'(\xi, \eta) = -k_1\eta + k_t\eta - X\eta + D, \quad \eta \to \infty, \quad (22c)$$

$$G(\xi, 0) = 0, \quad G(\xi, \eta) = Z(\eta - \alpha), \quad \eta \to \infty. \quad (23a, b)$$

Here

$$\Lambda_\phi = (2\xi/\phi)(d\phi/d\xi) = \frac{2\phi_s}{\phi r^j U_1(s,0)} \int_0^s U_1(s,0)\, r^{2j}\, ds \quad (24)$$

$$\alpha = \lim_{\eta \to \infty} (\eta - f) \quad (25a)$$

$$k_l = (2\xi)^{\frac{1}{2}} K/[r^j U_1(s,0)] \quad (25b)$$

$$k_t = (2\xi)^{\frac{1}{2}} j\cos\theta/[r^{2j} U_1(s,0)] \quad (25c)$$

$$D = U_2(s,0)/U_1(s,0) \quad (25d)$$

$$X = (2\xi)^{\frac{1}{2}} B_1(0)/U_1^2(s,0) \quad (25e)$$

$$Z = (2\xi)^{\frac{1}{2}} H_1'(0)/(t_w - H_1). \quad (25f)$$

The quantity Λ_{U_1} is termed as the principal velocity function [6], $\Lambda_{t_w - H_1}$ the principal thermal function, Λ_{k_l} the principal longitudinal curvature function, Λ_{k_t} the principal transverse curvature function and Λ_D the principal displacement speed function. These principal functions Λ_{k_l}, Λ_{k_t} and Λ_D are of fundamental importance in determining the structure of the second-order effects. Thus, it is the form of principal functions which differs from problem to problem in the bounday-layer flows. It is easily shown [6] that if two problems are similar in the sense of Reynolds similarity they have the same principal functions. The principal functions are as meaningful for a non-similar flow as for a similar flow, the main difference in the two applications being that while the principal functions remains constant in stream direction when flow is similar, they vary when flow is non-similar. The form of the principal function depends on both the flow and geometric configuration. In the present work we study the following general forms of the principal functions

$$\Lambda_{U_1} = \sum_{i=0} \beta_1 \xi^{mi}, \quad \Lambda_{t_w-H_1} = \sum_{i=0} \Lambda_i \xi^{mi} \quad (26a, b)$$

$$\Lambda_{k_l} = \sum_{i=0} \Lambda_{l_i} \xi^{mi}, \quad \Lambda_{k_t} = \sum_{i=0} \Lambda_{t_i} \xi^{mi} \quad (27a, b)$$

$$\Lambda_D = \sum_{i=0} \Lambda_{d_i} \xi^{mi}, \quad (27c)$$

where m is any positive or negative number, depending upon the flow. The function Λ_{U_1} for $m = 1$ and $\frac{1}{2}$ is used in Gortler's [6] study of first-order boundary-layer.

Now to study the second-order problem, described by equations (16)–(23) in addition to (26) and (27) we assume the following expansions

$$f = \sum_{i=0} f_i(\eta)\xi^{mi} \quad (28a)$$

$$g = \sum_{i=0} g_i(\eta)\xi^{mi} \quad (28b)$$

$$F = \sum_{i=0} (k_l F_i^{(l)} + k_t F_i^{(t)} + X F_i^{(v)} + D F_i^{(d)}) \quad (29a)$$

$$G = \sum_{i=0} (k_l G_i^{(l)} + k_t G_i^{(t)} + X G_i^{(v)} + D G_i^{(d)} + Z G_i^{(H)})\xi^{mi}. \quad (29b)$$

Substituting the above expansions (26a)–(29b) in equations (16)–(23) and collecting the coefficients of various powers of ξ^m, we get equations for successive approximations. In the present work only the solutions to leading terms will be presented. For terms of order ξ^m the governing equations and solutions for some cases are given in [5]. The leading terms for the first-order boundary layer are (the subscript '0' and superscripts l, t, v, etc., will be dropped for convenience of writing),

$$f''' + ff'' + \beta(1 - f'^2) = 0, \quad (30)$$

$$M(g; \Lambda) = 0, \quad (31)$$

$$f(0) = 0 = f'(0), \quad g(0) = 1, \quad (32a, b, c)$$

$$f'(\infty) = 1, \quad g(\infty) = 0, \quad (32d, e)$$

and the various second-order equations are as follows.

Longitudinal curvature

$$L(F, \Lambda_1) = [-\eta(\beta + 1)f''' + (\Lambda_1 + \beta - 1) \times (f'' + ff') + (2\beta + \Lambda_1)(\beta\eta + \alpha)]/(1 + \beta), \quad (33)$$

$$M(G, \Lambda + \Lambda_1) = \Lambda F'g - (1 + \Lambda_1)Fg' - \sigma^{-1}(\eta g')', \quad (34)$$

$$F(0) = 0 = F'(0), \quad G(0) = 0, \quad (35a, b, c)$$

$$F'(\eta) = -\eta, \quad \eta \to \infty; \quad G(\infty) = 0. \quad (35d, e)$$

Transverse curvature

$$L(F, \Lambda_t) = -\eta(2\beta + f''') + f'' + ff'' - \Lambda_t \eta f'^2 \quad (36)$$

$$M(G, \Lambda + \Lambda_t) = \Lambda F'g - (1 + \Lambda_t)Fg' - \sigma^{-1}(\eta g')' \quad (37)$$

$$F(0) = 0 = F'(0), \quad G(0) = 0, \quad (38a, b, c)$$

$$F'(\eta) = \eta, \eta \to \infty; \quad G(\infty) = 0. \quad (38d, e)$$

Vorticity

$$L(F, 1 - 2\beta) = -\alpha \quad (39)$$

$$M(G, \Lambda - 2\beta + 1) = \Lambda F'g - 2(1 - \beta)Fg' \quad (40)$$

$$F(0) = 0 = F'(0), \quad G(0) = 0, \quad (41a, b, c)$$

$$F'(\eta) = \eta, \quad \eta \to \infty; \quad G(\infty) = 0. \quad (41d, e)$$

Displacement speed

$$L(F, \Lambda_d) = -2\beta - \Lambda_d. \quad (42)$$

$$M(G, \Lambda + \Lambda_d) = \Lambda F'g - (1 + \Lambda_d)Fg', \quad (43)$$

$$F(0) = 0 = F'(0), \quad G(0) = 0, \quad (44a, b, c)$$

$$F'(\infty) = 1, \quad G(\infty) = 0. \quad (44d, e)$$

Stagnation enthalpy gradients

$$M(G, 1) = 0, \quad (45)$$

$$G(0) = 0, \quad G(\eta) = \eta - \alpha, \quad \eta \to \infty. \quad (46a, b)$$

Here L and M are the differential operators defined by

$$L(F, \lambda) = F''' + f F'' - (2\beta + \lambda)f'F' + (1 + \lambda)f''F \quad (47)$$

$$M(G,\lambda) = \sigma^{-1}G'' + f\,G' - \lambda f'G. \tag{48}$$

In general the above equations have to be integrated numerically to obtain skin friction τ and heat transfer at the wall q. The explicit expressions for these quantities are

$$\tau = \frac{r^j U_1(s,0)}{\sqrt{(2\xi)}}\,[R^{-\frac{1}{2}} f''(\xi,0) + R^{-1}\,F''(\xi,0) + \ldots] \tag{49}$$

$$= \sum_{i=0} (R^{-\frac{1}{2}}\tau_{1i} + R^{-1}\,\tau_{2i} + \ldots)\,\xi^{mi} \tag{50}$$

$$q = \frac{r^j U_1(s,0)}{\sqrt{(2\xi)}}(t_w - H_1(0))\,[R^{-\frac{1}{2}}\,g'(\xi,0) + R^{-1}\,G'(\xi,0) + \ldots] \tag{51}$$

$$= \sum_{i=0} (R^{-\frac{1}{2}}\,q_{1i} + R^{-1}\,q_{2i} + \ldots)\,\xi^{mi}. \tag{52}$$

Substituting the expansions (28) and (29) in above, the leading terms for the first-order boundary layer are

$$\tau_{11} = \frac{r^j U_1(s,0)}{\sqrt{(2\xi)}} f''(0), \tag{53a}$$

$$q_{11} = \frac{r^j U_1(s,0)}{\sqrt{(2\xi)}}\,[t_w - H_1(0)]\,g'(0) \tag{53b}$$

and for the second-order boundary layer are

$$\tau_{21} = K U_1(s,0) F''^{(l)}(0) + \frac{j\cos\theta}{r}\,U_1(s,0) \times F''^{(t)}(0) + r^j B_1'(0)\,F''^{(v)}(0) + \frac{U_2(s,0)U_1(s,0)}{\sqrt{(2\xi)}}\,r^j F''^{(d)}(0), \tag{54a}$$

$$q_{21} = -\sigma^{-1}\,H_1'(0)\,r^j U_1(s,0)\,G'^{(H)}(0) - \sigma^{-1}[t_w - H_1(0)]\,[K G'^{(l)}(0) + \frac{j\cos\theta}{r}\,G'^{(t)}(0) + \frac{r^j B_1'(0)}{U_1(s,0)} G'^{(v)}(0) + \frac{U_2(s,0)}{\sqrt{(2\xi)}}\,r^j G'^{(d)}(0)]. \tag{54b}$$

The various problems described by equations (30)–(46) are solved by Runge–Kutta procedure with Gill improvement (using step size $\Delta\eta = 0{\cdot}05$) on IBM 7044 Computer at Indian Institute of Technology, Kanpur (the basic initial values of $f''(0)$ of the Falkner–Skan equation (30) were taken from the tables of Smith [8]). However, before discussing the general results we wish to make a mention of the following special solutions.

4. SOME SPECIAL SOLUTIONS

(a) *For displacement speed problem*

It is known (e.g. [3, 7]) that the displacement speed problem (42)–(44) can be put in terms of first-order boundary layer functions for two special cases: viz: $\Lambda_d = 0$ and $\Lambda_d = \beta - 2$. For the first case we have

$$F^{(d)} = (\eta f' + f)/2 \tag{55a}$$

$$G^{(d)} = \eta g'/2 \tag{55b}$$

while for the second case we have

$$F^{(d)} = [(\beta - 1)\,\eta f' + f]/\beta \tag{56a}$$

$$G^{(d)} = [(\beta - 1)\,\eta g' + \Lambda g]/\beta. \tag{56b}$$

The solutions (56) show that the velocity and temperature profiles becomes singular at $\beta = 0$: a similar behaviour is observed in the other second-order effects (except for stagnation enthalpy gradient) for particular values of Λ_l.

(b) *Energy equation*

For suitable values of various parameters, the corresponding (second-order) contribution to heat transfer vanishes, e.g. when $\Lambda_l = -(1 + \Lambda)$ the energy equation (34) for the longitudinal curvature integrates to give

$$G^{(1)} = \int_0^\eta [\{\sigma\Lambda F^{(l)}(\eta_1) - \eta_1\}\,g(\eta_1)\exp\,\{\sigma \int_0^{\eta_1} f(\eta_2) - f(\eta)\,d\eta_2\}]\,d\eta_1 \tag{57}$$

leading to $G'^{(l)}(0) = 0$ for all β, σ and Λ. Similarly, it can be shown that

$$G'^{(t)}(0) = 0 \text{ for } \Lambda_t = -1 - \Lambda$$
$$G'^{(v)}(0) = 0 \text{ for } \Lambda = 2\beta - 2$$
$$G'^{(d)}(0) = 0 \text{ for } \Lambda_d = -1 - \Lambda$$
$$g'(0) = 0 \text{ for } \Lambda = -1.$$

5. DISCUSSION OF RESULTS

Our second-order momentum equations (33), (36) and (42) are similar to those of Werle and Davis [7], except for the choice of the parameters. The choice of parameters in the present work has the following advantage over π_2, π_3 and π_4 of [7]: the singularities observed in the second-order momentum solutions for a given β lies at Λ_{crit} which is same for all the three second-order effects, while in [7] $\pi_{2\,\text{crit}} \neq \pi_{3\,\text{crit}} = \pi_{4\text{crit}}$. Further, our calculations show the existence of an infinite number of singularities for negative values of the parameters for a given β. The authors of [7] pointed out only the first of these singularities and missed to note the others as they did not carry out the calculations up to sufficiently large values of the parameters.

In the discussion that follows we shall not include such aspects of the results of momentum equations as are essentially the same as those contained in [7].

The problems governing the second-order effects of longitudinal curvature, transverse curvature and displacement speed involve three parameters, viz Λ_{kl}, Λ_{kt} and Λ_D, while those of vorticity interaction and stagnation enthalpy gradients contain no new parameters. Hence, the latter two problems will be discussed first.

The stagnation enthalpy problem described by equations (45) and (46) is independent of the

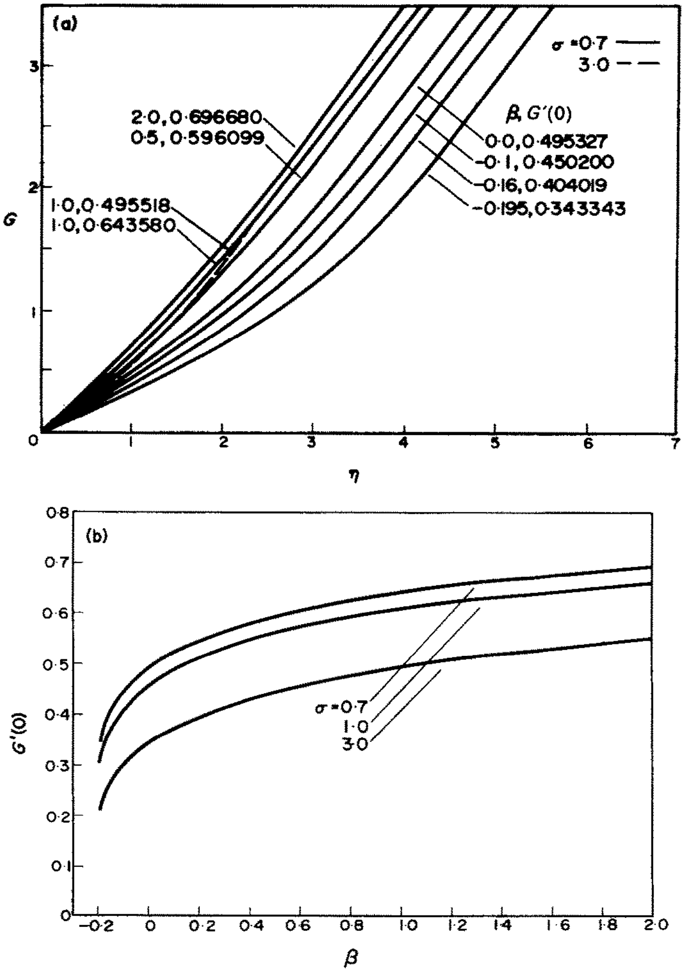


FIG. 2. Stagnation enthalpy solution: (a) temperature profiles, (b) heat transfer.

wall temperature parameter Λ, and depends only upon β and σ. It is well known that this effect does not contribute to velocity profiles [1]. The temperature profiles and heat-transfer rate for full range $\beta_s (= -0{\cdot}198838) \leqslant \beta \leqslant 2$ and $\sigma = 0{\cdot}7$, $1{\cdot}0$ and $3{\cdot}0$ are presented in the Figs. 2(a) and (b). The Fig. 2(b) shows that the second-order contribution to heat transfer increases when β increases for a given σ and decreases when increases for a given β. The sign of the second-order contribution to heat transfer is opposite to that of the first-order contribution; hence, the presence of the stagnation enthalpy gradient decreases the heat transfer. Furthermore, the corresponding second-order solutions are well-behaved right up to the *so called* separation point ($\beta = \beta_s$).

The problem of the vorticity interaction

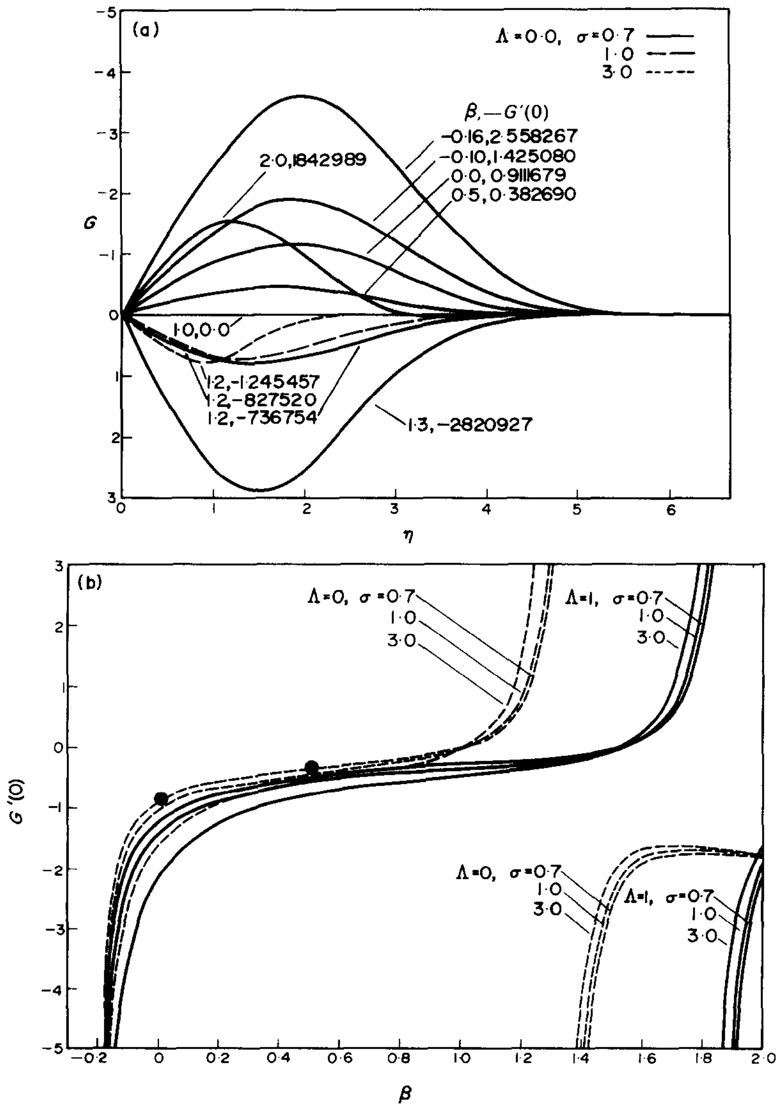


FIG. 3. Vorticity interaction solutions: (a) temperature profiles, (b) heat transfer, ● Van Dyke's.

governed by equations (39), (40) and (41), contains β, σ and Λ as parameters. As regards the solutions of the momentum equation (39), it is now available in [7] so we have shown the temperature profiles and heat-transfer rate in Figs. 3(a) and (b). For these solutions the values of β and σ are the same as in the case of enthalpy gradient, while $\Lambda = 0$, 1·0. The Fig. 3(b) shows that for a given β and Λ, there exists a value of $\beta = \beta_{cr}(\sigma, \Lambda)$ for which an infinite discontinuity is observed in heat transfer. For a given Λ, if σ increases the value of β_{cr} decreases. On the other hand if for a given σ, Λ increases the value of

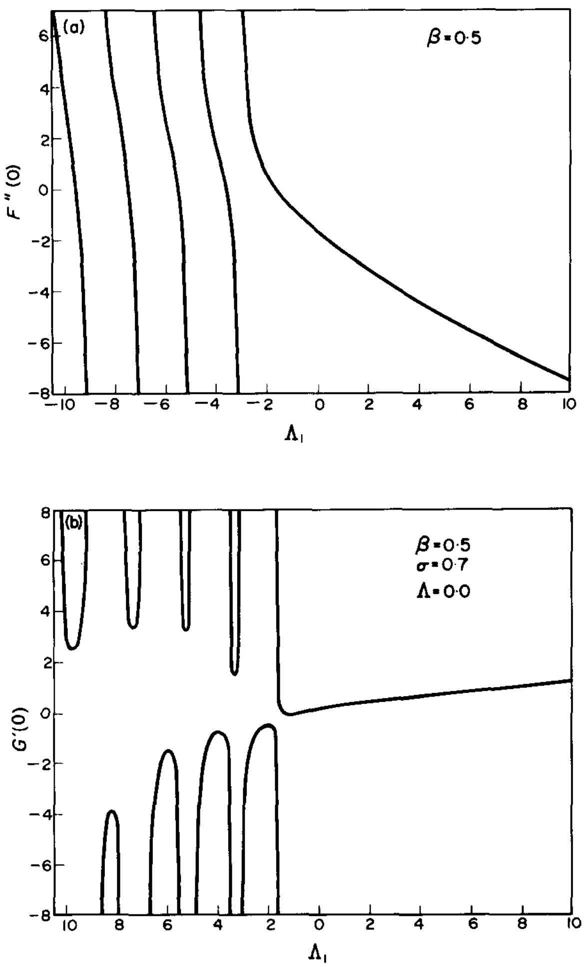


FIG. 4. Longitudinal curvature solutions: (a) skin friction, (b) heat transfer.

β_{cr} increases. Further, the exact solution (59) shows that the second-order contribution to heat transfer vanishes when $\beta = 1 + \Lambda/2$; this is demonstrated in Fig. 2 for the two cases: $\Lambda = 0$, $\beta = 1$ and $\Lambda = 1$, $\beta = 1·5$. The contribution of vorticity to heat transfer is in general, negative; however, for a small range of β (for example when $\sigma = 0·7$, $\Lambda = 0$ this range is $1 < \beta < 1·3$) it is positive.

The problem of longitudinal curvature is governed by equations (33)–(35) and involves a parameter Λ_1. Werle and Davis [7] have given a plot of skin friction vs. β with $\pi_3 (= \Lambda_1 + \beta - 1)$ as a parameter and have shown that the skin friction possesses an infinite discontinuity for some negative value of π_3 depending upon β. However their study does not show the nature of the contribution with respect to parameter π_3. We have displayed in Figs. 4(a) and (b) the skin friction and heat transfer vs. Λ_l with β as parameter. The Fig. 4(a) shows that the skin friction contribution is well-behaved for $\Lambda_l \geqslant 0$ while for $\Lambda_l < 0$ there exist a large number of infinite discontinuities but the essential nature of first) indicated in [7]. It is seen that for $\beta = 0·5$ the first of these infinite discontinuity lies near $\Lambda_l = -3$ (in agreement with [7]); all others follow towards left at an interval of -2 approximately. It can be seen that the effect of increase in β is only to displace, towards left, these infinite discontinuity but the essential nature of the second-order contribution is the same as shown in Fig. 4(a). The Fig. 4(b) shows that the heat transfer contribution possesses many more infinite discontinuities but the essential nature of friction (Fig. 4(a)), while for $\Lambda_l \geqslant 0$ (referred later as regular branch) heat transfer is well-behaved. Further, these discontinuities do not follow at regular intervals of Λ_l like in the skin friction, though all the points at which the skin friction is singular, so is the heat transfer. Thus for $\Lambda_l < 0$ the nature of the second order contribution is quite complex and efforts have been made, so far without success, to study the significance of this range of $\Lambda_l < 0$. However, for $\Lambda_l \geqslant 0$ it is clear that the second-order contribution to both the skin friction and heat transfer is of opposite sign compared to the respective quantities in the first-order; thus, the convex longitudinal surface curvature decreases the skin friction and heat transfer. Further, the second-order contributions to skin friction and heat transfer increases (in magnitude) monotonically with Λ_l. To study the detailed effects of σ and Λ on the second-order heat transfer vs. β we select a sufficiently representative value of $\Lambda_l(=0)$ on regular branch $\Lambda_l \geqslant 0$. Further, this particular value of $\Lambda_l(=0)$ corresponds to the classical similarity studied in [3] and by Narasimha and Ojha [9]. Temperature profiles for this case (of $\Lambda_l = 0$) for a constant wall temperature ($\Lambda = 0$) are shown in Fig. 5(a). Some representative results for heat transfer are shown in Fig. 5(b). This figure shows that for a given σ and Λ an increase of β decreases the second-order contribution to the heat transfer while an increase in σ (alternately Λ) for a given Λ (given σ) and β increases the second-order contribution.

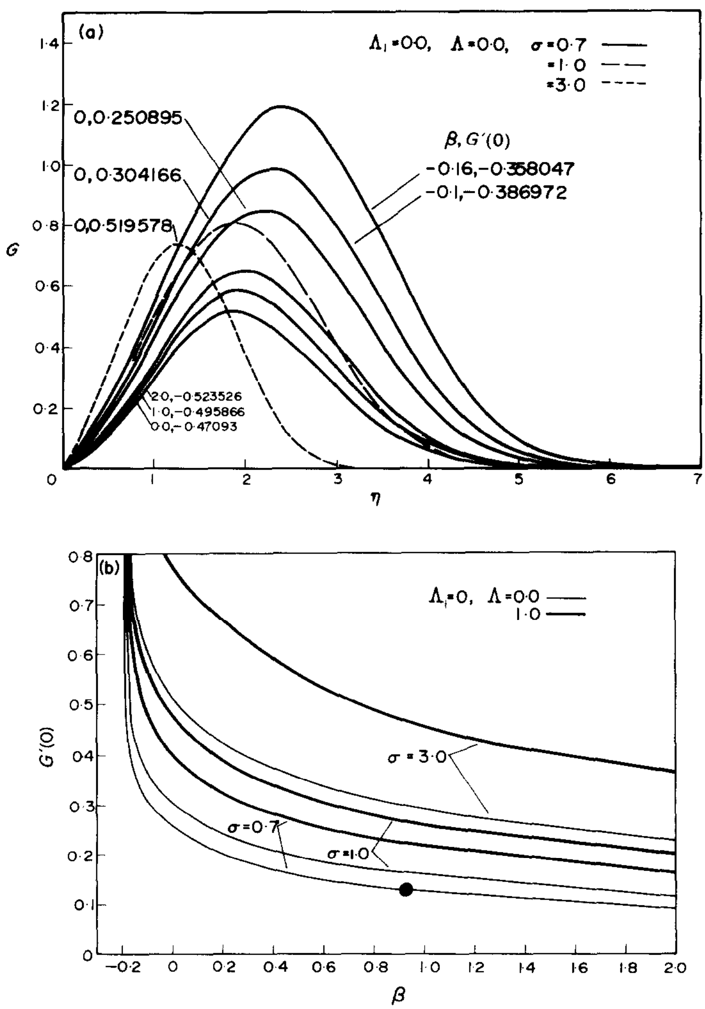


FIG. 5. Longitudinal curvature solutions: (a) temperature profiles, (b) heat transfer; ● Van Dyke's.

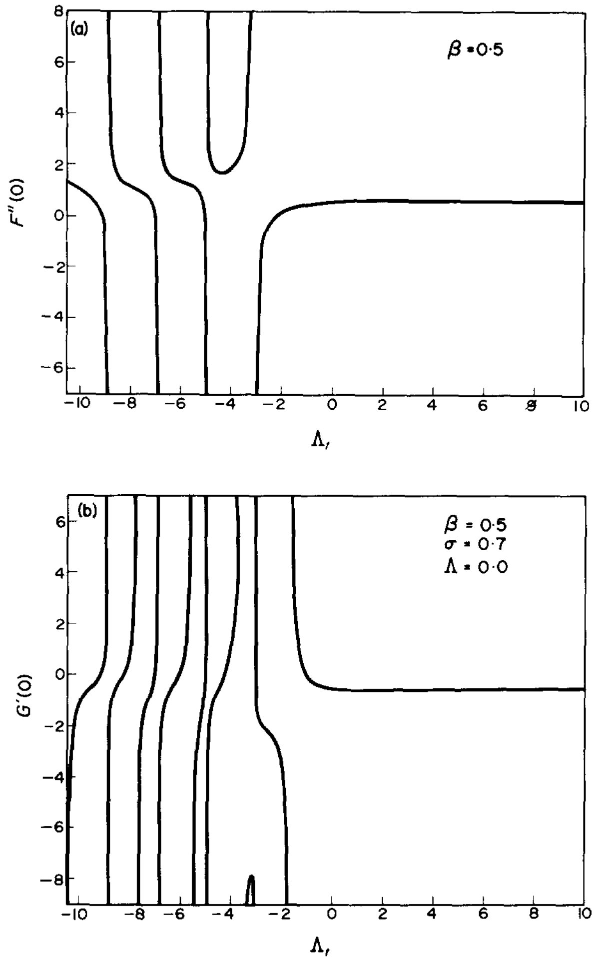


FIG. 6. Transverse curvature solutions: (a) skin friction (b) heat transfer.

The second-order contribution of transverse curvature to the skin friction and the heat transfer are shown in Figs. 6(a) and (b). In view of the discussion already given the results are self explanatory and need no additional comments regarding the infinite discontinuities. For $\Lambda_t \geqslant 0$ it is clear that the transverse curvature increases the skin friction and heat transfer. The second-order contributions increase monotonically with Λ_t. Temperature profiles for the case $\Lambda_t = 0$ with constant wall temperature for various values of β and σ are shown in Fig. 7(a). The Fig. 7(b) displays some representative heat-transfer rates vs. β with σ and Λ as parameters, on regular branch. The figure shows that the second-order contribution to the heat transfer decreases (in magnitude) if β increases for a given σ and Λ. On the other hand as σ (alternatively Λ) increases for a given Λ (given σ) and β the contribution increases.

The Figs. 8(a) and (b) show the contribution to skin friction and heat transfer respectively,

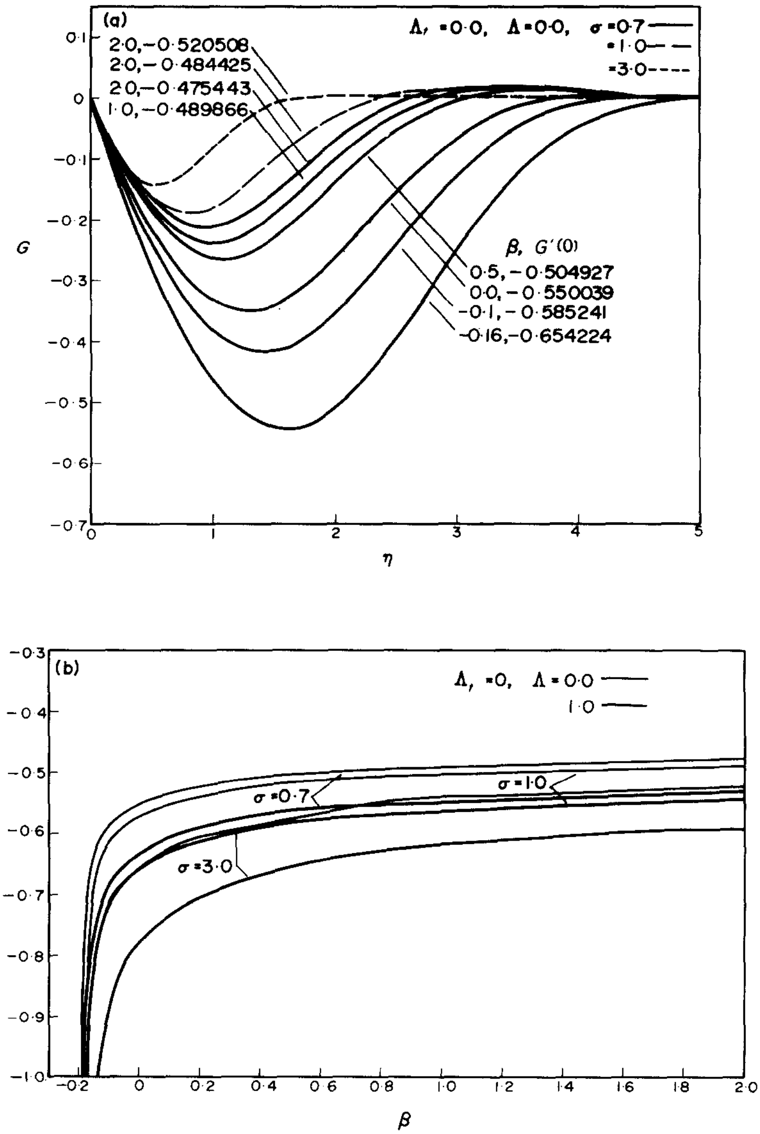
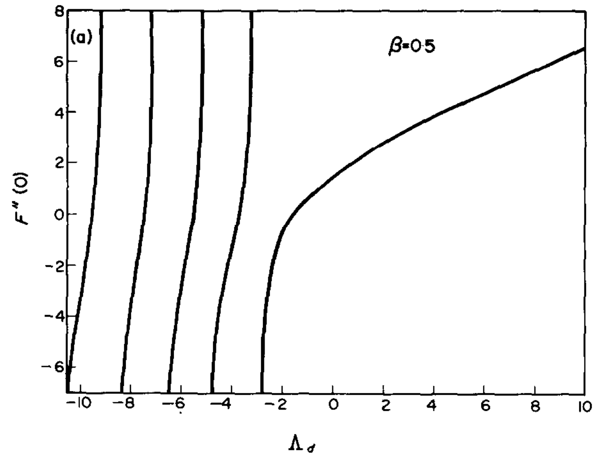


FIG. 7. Transverse curvature solutions: (a) temperature profiles, (b) heat transfer.

due to displacement speed. Here, too, no additional comments are needed. For $\Lambda_d \geqslant 0$ it is evident from the figures that the second-order contribution to the heat transfer is of the same sign as the first-order; thus, for positive displacement speed $[U_2(s, 0) > 0]$ this effect increases the skin friction and heat transfer. The results for $\Lambda_d = 0$ are studied in detail. Figure 9(a) shows the temperature profiles, while the Fig. 9(b) 9(b) shows the heat-transfer rate. The latter shows that the contribution to the heat transfer increases (in magnitude) as β increases for a



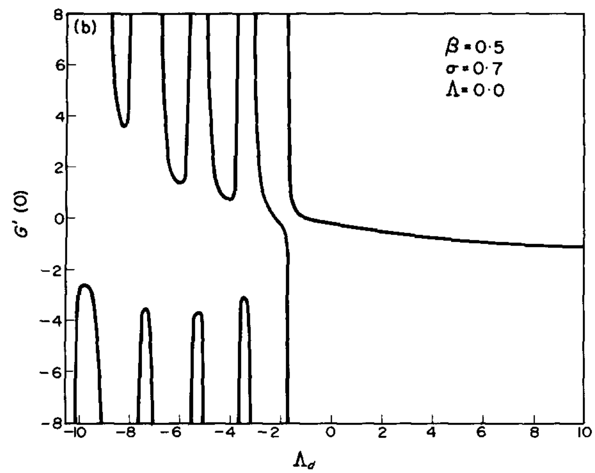


FIG. 8. Displacement speed solutions: (a) skin friction (b) heat transfer.

given σ and Λ. Further, as σ (alternately Λ) increases for a given Λ (given σ) and β the second-order contribution increases.

Thus, we see that a large number of infinite discontinuities occurs in each of the longituninal curvature, transverse curvature and displacement speed problems. The number of such discontinuities in the heat transfer equation is much larger than in the momentum equation. For a given β and singularity the discontinuities in all momentum equations lie at Λ_{crit_1} $(= \Lambda_{l_{\text{crit}}} = \Lambda_{t_{\text{crit}}} = \Lambda_{d_{\text{crit}}})$ while for the heat transfer equation they lie at Λ_{crit_2} $(= \Lambda_{l\text{crit}} = \Lambda_{t\text{crit}} = \Lambda_{d\text{crit}})$.

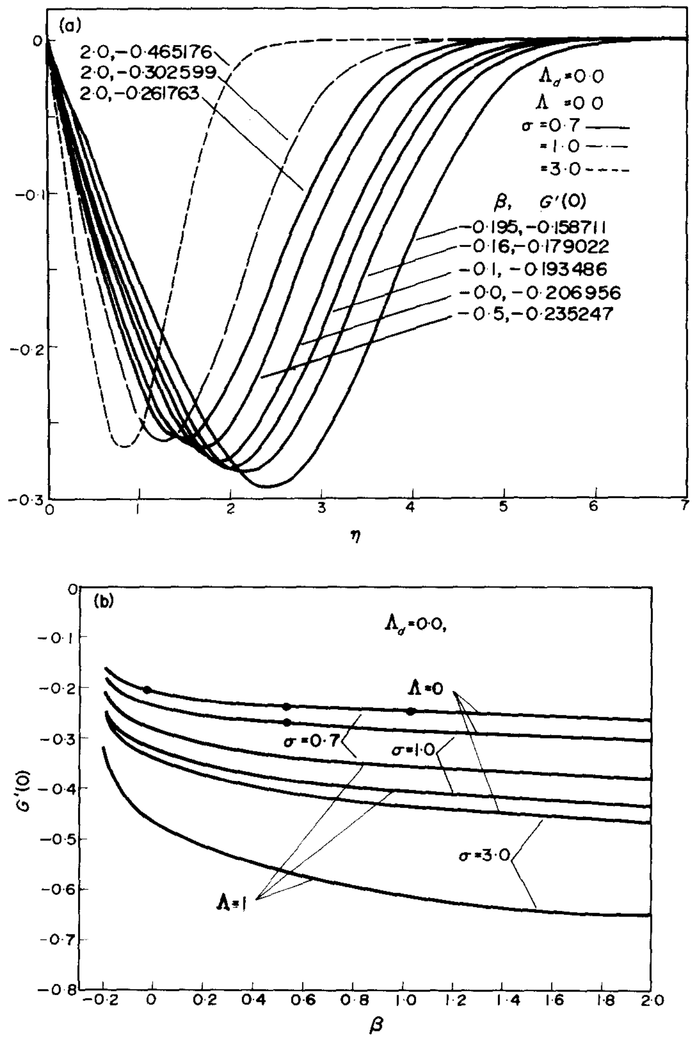


FIG. 9. Displacement speed solutions: (a) temperature profiles (b) heat transfer, ● Van Dyke.

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SOLUTIONS EN SELF-SIMILITUDE DES ÉCOULEMENTS AVEC COUCHE LIMITE POUR UN FLUIDE INCOMPRESSIBLE AVEC TRANSFERT THERMIQUE

Résumé—On étudie les solutions en self-similitude des équations du second ordre de la couche limite incompressible contenant des termes dus à la courbure longitudinale, la courbure transversale, la vorticité externe, aux gradients d'enthalpie d'arrêt. On présente des solutions numériques et quelques solutions analytiques aussi bien pour le frottement pariétal que pour le flux thermique pariétal.

ÄHNLICHE LÖSUNGEN DER GRENZSCHICHTSTRÖMUNGEN ZWEITER ORDNUNG BEI EINEM INKOMPRESSIBLEN MEDIUM MIT WÄRMEÜBERGANG

Zusammenfassung—Es werden ähnliche Lösungen der Gleichungen für inkompressible Grenzschichten zweiter Ordnung untersucht, die Glieder für Längskrümmung, Querkrümmung, äussere Wirbel, Verdrängungsgeschwindigkeit und Stauenthalpie enthalten. Neben numerischen Lösungen werden auch mehrere Lösungen in geschlossener Form sowohl für die Wandreibung als auch für den Wärmestrom an der Wand angegeben.

АВТОМОДЕЛЬНЫЕ РЕШЕНИЯ ТЕЧЕНИЙ В ПОГРАНИЧНОМ НЕСЖИМАЕМОМ СЛОЕ ВТОРОГО ПОРЯДКА С ТЕПЛООБМЕНОМ

Аннотация—Изучены автомодельные решения уравнений несжимаемого пограничного слоя второго порядка, содержащие члены продольной и поперечной кривизны, внешней завихренности, изменения толщины вытеснения и энтальпии в критической точке. Представлены численные решения, а также решения в замкнутом виде для поверхностного трения и интенсивности теплообмена.